MOCK TEST – 1

SUBJECT: MATHEMATICS

ANSWERS: COMMON ENTRANCE TEST 2012

Solutions

1.Ans(3). Here $a = 2n$ or $a = 2n + 1$.

$\therefore\ a^2 = 4n^2$ (this is divisible by 4) or $a^2 = (2n + 1)^2 = 4n^2 + 4n + 1$.
(This leaves the remainder 1 when divided by 4)

2.Ans(3). By property, x and y are relatively prime.

3.Ans(1).$180 = 2^2 3^2 5^1$. $\therefore$ the sum of all the positive divisors of 180 excluding 1 and itself is $= \frac{2^3 - 1}{2 - 1} . \frac{3^3 - 1}{3 - 1} . \frac{5^2 - 1}{5 - 1} - 181 = 7.13.6 - 181 = 365$

4.Ans(4). The given determinant $= \begin{vmatrix} \sin(\pi) & \sin B & \cos z \\ -\sin B & 0 & \tan A \\ \cos(\pi - z) & -\tan A & 0 \end{vmatrix}$

$= \begin{vmatrix} 0 & \sin B & \cos z \\ -\sin B & 0 & \tan A \\ -\cos z & -\tan A & 0 \end{vmatrix} = 0$

$\because$ the corresponding matrix is skew-symmetric.

5.Ans(2). Determinant of the matrix of the cofactors is same as the determinant of the adjoint of the matrix, which is equal to $11^2 = 121$. ($\because |adj\ A| = |A|^{n-1}$)

6.Ans(1). $adj\ (A^{-1}) = (adj\ A)^{-1} = \frac{A}{|A|}$. Now,

$|A| = 1.(9 + 2) + 1.(6 - 1) + 2.(-4 - 3) = 11 + 5 - 14 = 2$.

$\therefore\ (adj\ A)^{-1} = \frac{1}{2}A$.

7.Ans(3). Since the matrix is triangular the diagonal elements are the Eigen values. By the given condition, $x - a = x - c = x - f$. $\therefore\ a = c = f$.

8. Ans(2). $|\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}| cos\ \theta = 1 + 1 + 2cos\ \theta$

$= 2\ (1 + cos\ \theta) = 2.2cos^2\ \frac{\theta}{2}$. $\therefore\ |\vec{a} + \vec{b}| = 2cos\ \frac{\theta}{2}$.

This is less than one if $cos\ \frac{\theta}{2} < \frac{1}{2}$.

This is when $\pi / 3 < \theta/2 < \pi / 2 \Rightarrow 2\pi/3 < \theta/2 < \pi$.

9. Ans(4). Magnitudes of the vectors are $\sqrt{49 + 121 + 1} = \sqrt{171}$,
$\sqrt{25 + 9 + 4} = \sqrt{38}$ and $\sqrt{144 + 64 + 1} = \sqrt{209}$.
Here $171 + 38 = 209$.

10. Ans(2). $\vec{a} \times \vec{b} - \vec{a} \times \vec{c} = \vec{c} \times \vec{d} - \vec{b} \times \vec{d}$

$\Rightarrow \vec{a} \times (\vec{b} - \vec{c}) = (\vec{c} - \vec{b}) \times \vec{d}$

$\Rightarrow \vec{a} \times (\vec{b} - \vec{c}) - (\vec{c} - \vec{b}) \times \vec{d} = \vec{o}$
$\Rightarrow \vec{a} \times (\vec{b} - \vec{c}) - \vec{d} \times (\vec{b} - \vec{c}) = \vec{o}$
$\Rightarrow (\vec{a} - \vec{d}) \times (\vec{b} - \vec{c}) = \vec{o}$
$\Rightarrow \vec{a} - \vec{d}$ is parallel to $\vec{b} - \vec{c}$.

11. Ans(1). Observe that $\vec{a} \times (\vec{b} + \vec{c}) + \vec{b} \times (\vec{c} + \vec{a}) + \vec{c} \times (\vec{a} + \vec{b})$
$= (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c}) + (\vec{b} \times \vec{c}) + (\vec{b} \times \vec{a}) + (\vec{c} \times \vec{a}) + (\vec{c} \times \vec{b}) = \vec{o}$.
$\therefore$ the value of the determinant corresponding to the given scalar triple product is zero.

12. Ans(4). 'Addition' is a binary operation on N. Under $+$, N has no identity element
Under this operation, G is an abelian group. 'subtraction' is a binary operation on the set Z. But $a - (b - c) \neq (a - b) - c$.
The operation $*$ defined on N by $a * b = a^b + b^a$ is both binary and commutative. But this is not associative (verify this).
If $*$ is a binary operation defined on the singleton set $G = \{a\}$ then $a * a = a$.

13. Ans(4). $2^{n^2} = 2^{2^2} = 16$.

14. Ans(2). Here identity is 3. $\therefore$ inverse of 3 is 3 itself.

15.Ans (1). $\sqrt{1 + \sin x} = \sqrt{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} + 2.\sin\frac{x}{2}.\cos\frac{x}{2}}$
$= \sqrt{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2} = \cos\frac{x}{2} + \sin\frac{x}{2}$.
Similarly $\sqrt{1 - \sin x} = \cos\frac{x}{2} - \sin\frac{x}{2}$.
(Since $0 < x < \frac{\pi}{2}$ we have, $\cos\frac{x}{2} > \sin\frac{x}{2}$).
$\therefore \cot^{-1}\left\{\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right\} = \cot^{-1}\frac{\cos\left(\frac{x}{2}\right) + \sin\left(\frac{x}{2}\right) + \cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right) + \sin\left(\frac{x}{2}\right) - \left(\cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right)\right)}$
$= \cot^{-1}\cot\frac{x}{2} = \frac{x}{2}$.

16.Ans (3). By definition, $-1 \le x - 1 \le 1$. $\therefore 0 \le x \le 2$.

17.Ans(1). The given equation is $\frac{1 + \tan\theta}{1 - \tan\theta} + \frac{1 - \tan\theta}{1 + \tan\theta} = 4$
$\Rightarrow (1 + \tan\theta)^2 + (1 - \tan\theta)^2 = 4(1 + \tan\theta)(1 - \tan\theta)$
$\Rightarrow 2(1 + \tan^2\theta) = 4(1 - \tan^2\theta)$
$\Rightarrow 3.\tan^2\theta = 1 \Rightarrow \tan\theta = \pm\frac{1}{\sqrt{3}} \Rightarrow \theta = n\pi \pm \frac{\pi}{6}$.

18.Ans(2). $\frac{1}{\sqrt{6} + \sqrt{2}} = \frac{\sqrt{3} - 1}{2\sqrt{2}}$ and $\frac{1}{\sqrt{6} - \sqrt{2}} = \frac{\sqrt{3} + 1}{2\sqrt{2}}$.
$\therefore \left(\frac{1}{\sqrt{6} + \sqrt{2}} + i\frac{1}{\sqrt{6} - \sqrt{2}}\right)^{10} = (\cos 75° + i\sin 75°)^{10} = \cos 750° + i\sin 750°$
$= \cos 30° + i\sin 30° = \frac{\sqrt{3}}{2} + i\frac{1}{2}$.

19.Ans(4). $\frac{(\sqrt{3}+i)^{4n+1}}{(1-i\sqrt{3})^{4n}} = \frac{(\sqrt{3}+i)^{4n}(\sqrt{3}+i)}{(1-i\sqrt{3})^{4n}} = \frac{\left(i(1-i\sqrt{3})\right)^{4n}(\sqrt{3}+i)}{(1-i\sqrt{3})^{4n}} = \sqrt{3}+i.$

$\therefore$ argument $= tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}.$

20.Ans(3). $x^2+y^2 = \frac{5^2+2^2}{7^2+1^2} = \frac{29}{50}.$

21.Ans(1). Let OP and OQ are the tangents drawn from the origin to the given circle (draw a figure). Now, center $C \equiv (7,1)$ and the radius $CP = 5$.
Also $OC = \sqrt{49+1} = 5\sqrt{2}$.
If '2A' is the angle between the 2 tangents then $sin\,A = \frac{CP}{OC} = \frac{5}{5\sqrt{2}} = \frac{1}{\sqrt{2}}.$
$\therefore A = \pi/4 \therefore 2A = \pi/2.$

22.Ans(2). The circle, $x^2+y^2+ax = 0$ has its center on the x-axis and passes through the origin. The circle, $x^2+y^2 = c^2,\ c > 0$ is centered at the origin. $\therefore$ 2 circles intersect at two points.

23.Ans(3). $16x = 4 - 9 \Rightarrow 16x = 5.$

24.Ans(4). Length of the perpendicular from the center to the given line,
$\left|\frac{0+0-1}{\sqrt{h^2+k^2}}\right| = a \Rightarrow \frac{1}{h^2+k^2} = a^2 \Rightarrow h^2+k^2 = \frac{1}{a^2}.$
$\therefore$ the locus of (h,k) is $x^2+y^2 = \frac{1}{a^2}.$

25.Ans (2). Slope of the normal is tan 45° $= 1$. $\therefore$ Slope of the tangent is -1.
$\therefore$ the required point is $\left(\frac{a}{m^2}, \frac{2a}{m}\right) \equiv \left(\frac{1}{1}, \frac{2}{-1}\right) \equiv (1, -2).$

26.Ans (3). Point of intersection of the perpendicular tangents lies on the directrix, $x = -a = -3$. Equation of the tangent having slope $3/2$ is $y = (3/2)x + \frac{3}{3/2}$. By taking $x = -3$ we get, $y = -5/2$.

27.Ans (4). $(2, 3)$. This is by the definition of parametric form of a parabola.

28.Ans (1). $16x^2+9y^2 = 144 \Rightarrow \frac{x^2}{9}+\frac{y^2}{16} = 1$. $\therefore$ the auxiliary circle is $x^2+y^2 = 16$.

29.Ans (4). $fog(x)+gof(x) = e^{\log x}+log\,(e^x) = x+x = 2x.$
$\therefore$ the derivative of $fog+gof$ is 2.

30.Ans (3). $sec\,y = x+y \Rightarrow sec\,y.tan\,y = \frac{dx}{dy}+1.\ \therefore \frac{dt}{dy} = sec\,y.tan\,y$
$\Rightarrow t = sec\,y.$

31.Ans (2). Let $\sin x = \tan y$. Then $f(x) = sec^{-1}\left(\frac{1+tan^2y}{1-tan^2y}\right)$
$= sec^{-1}(sec\,2y) = 2y = 2tan^{-1}(sin\,x).\ \therefore f'(x) = \frac{2.cos\,x}{1+sin^2x}$
$\therefore f'\left(\frac{\pi}{4}\right) = \frac{2(1/\sqrt{2})}{1+(1/2)} = \frac{2\sqrt{2}}{3}.$

32.Ans (1). $\frac{dy}{dx} = \frac{g'}{f'}$. $\therefore$ $\frac{d^2y}{dx^2} = \frac{f'.g'' - g'.f''}{(f')^2} \cdot \frac{1}{f'}$.

33.Ans (4). xy $= 1 \Rightarrow \frac{dy}{dx} = -\frac{y}{x}$. $\therefore$ slope of the normal $\frac{x}{y} = -\frac{a}{b}$.
Now xy $= 1 \Rightarrow$ both x and y have the same sign.
$\therefore$ $a > 0,\ b < 0$ is the correct choice.

34.Ans (3). $x^2 + y^2 = 4x \Rightarrow \frac{dy}{dx} = \frac{4 - 2x}{2y}$. $\therefore$ slope at $(2, 2)$ is $= 0$. Here the tangent is parallel to the x-axis.
$x^2 + y^2 = 8 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$. $\therefore$ slope at $(2, 2)$ is $= -1$.
Therefore the angle between the curves is $= \frac{\pi}{4}$ (Draw the figure).
$\therefore$ $p = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$.

35.Ans.(1). The perimeter is least when the rectangle is a square.
Area $= 100 \Rightarrow$ length of the side is 10.
Thus the least perimeter is $= 4.10 = 40$.

36.Ans.(2). $\frac{ds}{dt} = 6t^2 - 18t + 15$ and $\frac{d^2s}{dt^2} = 12t - 18$. Now $12t - 18 = 0$
$\Rightarrow t = 3/2$.

37.Ans.(1). Derivative of $log(x^2 + 1)$ is $\frac{2x}{x^2 + 1}$.
$\therefore \int e^x \left\{log(x^2 + 1) + \frac{2x}{x^2 + 1}\right\} dx = e^x.log\,(x^2 + 1)$ $\therefore$ $(b,\ c) \equiv (2,\ 1)$

38.Ans.(4). $\int \sqrt{\frac{\cos x - \cos^3 x}{1 - \cos^3 x}}\, dx = \int \sqrt{\frac{\cos x. \sin^2 x}{1 - \cos^3 x}}\, dx = \int \frac{\sqrt{\cos x}}{\sqrt{1 - (\cos^{3/2} x)^2}} \sin x. dx.$

Put $\cos^{3/2}x = y$. Then $-\frac{3}{2}\cos^{1/2}x. \sin x. dx = dy$.
$\therefore$ above integral $= \int \frac{-1}{\sqrt{1 - t^2}} \frac{2}{3} dy = \frac{2}{3}.\cos^{-1}y$.

39.Ans.(2). $\int \cos x. cosec^2 x\ dx = \int \cot x. cosec\, x\,.dx = -\,cosec\, x$.

40.Ans:(3). If $0 < x < \pi$, $y = x.\sin x$ is positive and if $\pi < x < 2\pi$, $y = x.\sin x$ is negative. $\therefore$ the area of the region
$= \int_0^{\pi} x.\sin x\ dx + \int_{\pi}^{2\pi} -x.\sin x\ dx$
$= [-x.\cos x + \sin x]_0^{\pi} - [-x.\cos x + \sin x]_{\pi}^{2\pi}$
$= -\pi.\cos\pi - (-2\pi\cos 2\pi + \pi\cos\pi) = 4\pi$.

41.Ans:(4). $\int_0^{3/2} [x^2]\, dx = \int_0^1 0\, dx + \int_1^{\sqrt{2}} 1\, dx + \int_{\sqrt{2}}^{3/2} 2\, dx$
$= 0 + (\sqrt{2} - 1) + \left(2(3/2) - 2\sqrt{2}\right) = 2 - \sqrt{2}$.

42.Ans:(3). Put $tan^{-1}x = t$. Then $dx = sec^2 t. dt$.
Also when $x = 0,\ t = 0$ and when $x = 1,\ t = \pi/4$.
$\therefore I_1 = \int_0^{\pi/4} \frac{t}{\tan t} sec^2 t. dt = \int_0^{\pi/4} \frac{t}{\sin t.\cos t} dt = \int_0^{\pi/4} \frac{2t}{\sin 2t} dt$.
By taking $2t = x$, we get, $2dt = dx$.

Also when $t = 0, x = 0$ and $t = \pi/4, x = \pi/2$.

$\therefore I_1 = \int_0^{\pi/4} \frac{2t}{\sin 2t} dt = \int_0^{\pi/2} \frac{x}{\sin x} \frac{dx}{2} = I_2$.

43.Ans:(2). By replacing x by $1 - k + k - x = 1 - x$, we get,

$I_1 = \int_{1-k}^{k} (1 - x). f((1-x)x)\, dx = I_2$.

44.Ans:(1). $y = \cos bx \Rightarrow \cos^{-1} y = bx \Rightarrow \frac{-1}{\sqrt{1 - y^2}} y_1 = b$.

By squaring we get, $\frac{{y_1}^2}{1 - y^2} = b^2 = \left(\frac{\cos^{-1} y}{x}\right)^2$.

45.Ans:(4). Put $x + y = v$. Then $1 + \frac{dy}{dx} = \frac{dv}{dx}$.

$\therefore \frac{dy}{dx} = \cos(x + y) \Rightarrow \frac{dv}{dx} - 1 = \cos v \Rightarrow \frac{dv}{dx} = 2.\cos^2\left(\frac{v}{2}\right)$

$\Rightarrow \frac{1}{2} \sec^2\left(\frac{v}{2}\right).dv = dx$. By integrating we get, $\tan\frac{v}{2} = x + c$

$\Rightarrow v = 2\tan^{-1}(x + c) \quad \Rightarrow x + y = 2\tan^{-1}(x + c)$.

When $x = 0$ and $y = \pi/2$ we get $c = 1$.

Thus the particular solution is $y = 2\tan^{-1}(x + 1) - x$.

46. Ans(4). n^{th} term $= \frac{2n + 1}{n^2(n + 1)^2} = \frac{(n + 1)^2 - n^2}{n^2(n + 1)^2} = \frac{1}{n^2} - \frac{1}{(n + 1)^2}$.

The sum to n terms is

$= \left(\frac{1}{1^2} - \frac{1}{2^2}\right) + \left(\frac{1}{2^2} - \frac{1}{3^2}\right) + \left(\frac{1}{3^2} - \frac{1}{4^2}\right) + \ldots + \left(\frac{1}{n^2} - \frac{1}{(n + 1)^2}\right)$

$= 1 - \frac{1}{(n + 1)^2}$. $\therefore 1 - \frac{1}{(n + 1)^2} = \frac{624}{625} = 1 - \frac{1}{625}$

$\Rightarrow n + 1 = 25 \Rightarrow n = 24$.

47. Ans(3). $\log_{100} 5 = \frac{\log_4 5}{\log_4 100} = \frac{k}{\log_4 25 + \log_4 4} = \frac{k}{2\log_4 5 + 1} = \frac{k}{2k + 1}$.

48.Ans(2). $2[\,{}^nC_1 + 2\{\,{}^nC_2 + 2({}^nC_3 + 2(\ldots + 2.{}^nC_n\,))\}]$

$= {}^nC_1.2 + {}^nC_2.2^2 + {}^nC_3.2^3 + \ldots + {}^nC_n.2^n$

$= (1 + 2)^n - {}^nC_0 = 3^n - 1$. Now $3^n - 1 = 242 \Rightarrow 3^n = 243 \Rightarrow n = 5$.

49.Ans(1). $\sim(p \leftrightarrow q) \equiv \sim\{(p \to q) \wedge (q \to p)\}$

$\equiv \sim(p \to q) \vee \sim(q \to p)$

$\equiv \{(p \wedge \sim q) \vee \sim(q \to p)\}$.

50.Ans(2). f(x) is defined only when $|x| - x > 0$.

This is when $|x| > x$. $\therefore x < 0$.

51.Ans(3). $\sum \frac{\alpha}{\beta\gamma\delta} = \sum \frac{\alpha^2}{\alpha\beta\gamma\delta} = \frac{1}{\alpha\beta\gamma\delta}.((\sum\alpha)^2 - 2\sum\alpha\beta) = \frac{1}{5}((2)^2 - 2.3) = \frac{-2}{5}$.

52.Ans(1). Here $D \equiv (a/2, 0)$ and $E \equiv (a/2, b/2)$.

Now slopes of AD and BE are $\frac{-b}{a/2}$ and $\frac{b/2}{a/2}$.

Product of the slopes $= -1 \Rightarrow \frac{-b^2/2}{a^2/4} = -1 \Rightarrow a^2 = 2b^2 \Rightarrow a = \pm\sqrt{2}b$.

53.Ans(1). The lines represented by the given equation are perpendicular (here $a + b = 0$). ∴ diagonals are perpendicular.

54.Ans(2). By squaring and adding the equations we get $x^2 + y^2 = a^2 + b^2$. The point $(2, 3)$ lies on the locus. ∴ $a^2 + b^2 = 2^2 + 3^2 = 13$.

55.Ans(4). $cos\ 2\theta + 2cos\ \theta = 2cos^2\theta - 1 + 2cos\ \theta$

$= 2(cos^2\theta + cos\ \theta + \frac{1}{4}) - \frac{1}{2} - 1 = 2\left(\cos\theta + \frac{1}{2}\right)^2 - \frac{3}{2}$.

The greatest value is $2\left(1 + \frac{1}{2}\right)^2 - \frac{3}{2} = 3$ and least value is $-\frac{3}{2}$.

Since $3 \le 2\left(\cos\theta + \frac{1}{2}\right)^2 - \frac{3}{2} \le -\frac{3}{2}$ The expression can be zero or one for some θ.

56.Ans(3). By using the half angle formulae for $tan\left(\frac{A}{2}\right)$ and $tan\left(\frac{B}{2}\right)$ we establish that $(s - a).tan\left(\frac{A}{2}\right) = (s - b).tan\left(\frac{B}{2}\right)$.

57.Ans(2). sin 12°.sin24°.sin48°.sin84°

$= \frac{1}{\sin 72°.\sin 36°}$(sin 12°.sin 72°.sin48°).(sin24°.sin36°. sin84°)

$= \frac{1}{\sin 72°.\sin 36°}.\frac{1}{4}(\sin 3(12°)).\frac{1}{4}.(\sin 3(24°)) = \frac{1}{16}$.

(Note : $\sin\theta.\sin(60° + \theta).\sin(60° - \theta) = \frac{1}{4}.\sin 3\theta$)

58.Ans(3). By using the formula $s = r\theta$ we conclude that $\frac{r_1}{r_2} = \frac{\theta_2}{\theta_3} = \frac{75}{60} = \frac{5}{4}$.

59. Ans(1). $\lim_{x \to 0} f(x) = \lim_{x \to 0}\left(x.\sin\left(\frac{1}{x}\right)\right) = \lim_{x \to 0}(x).\lim_{x \to 0}\left(\sin\left(\frac{1}{x}\right)\right)$

$= 0.$(a real number) $= 0 = f(0)$.

60. Ans(4). $\lim_{x \to 0}\left(\frac{3^{2x} - 2^{2x}}{x}\right)^2 = \lim_{x \to 0}\left(\frac{3^{2x} - 1 - (2^{2x} - 1)}{x}\right)^2$

$= \lim_{x \to 0}\left(\frac{3^{2x} - 1}{x} - \frac{2^{2x} - 1}{x}\right)^2 = (\log_e 9 - \log_e 4)^2 = (log(9/4))^2$.