1. Define a binary operation
   Ans: Let \( S \) be a nonempty set. \( * \) is said to be binary operation on \( S \) if
   \[ \forall a, b \in S, \quad a * b \in S \quad \text{and is unique} \]

2. \( a * b = \sqrt{ab} \), Where \( a \) and \( b \) are the elements of \( R \). Is \( * \) a binary operation on \( R \)?
   Ans: Let \( a = 2, b = -7 \) then \( a * b = 2 * (-7) = \sqrt{-14} \notin R \)
   \( \therefore * \) is not a binary operation.

3. \( a * b = \sqrt{ab} \), Where \( a \) and \( b \) are the elements of \( C \). Is \( * \) a binary operation on \( C \)?
   Ans: Let \( a = 1, b = i \) then \( a * b = 1 * i = \sqrt{i} \in C \) but \( \sqrt{i} \) is not unique
   \( \therefore * \) is not a binary operation.

4. In the set of all of natural numbers \( N \), \( * \) is defined by \( a * b = 3a - 4b \). Is \( * \) a binary operation.
   Ans: Let \( a = 2, b = 3 \) then \( a * b = 2 * 3 = 3(2) - 4(3) = 6 - 12 = -6 \notin N \)
   \( \therefore * \) is not a binary operation.

5. In the set of nonnegative integers \( * \) is defined by \( a * b = a^b \). Verify whether \( * \) is a binary operation or not.
   Ans: Let \( a = 0, b = 0 \) then \( a * b = 0 * 0 = 0^0 \) is not defined.
   \( \therefore * \) is not a binary operation.

6. If \( * \) is defined by \( a * b = 1 + ab \), show that it is a b.o. on \( Q \).
   Ans: Let \( a, b \in Q \). Product \( a \cdot b \) of two rational numbers is again a rational number and \( 1 + ab \) is also a rational number.
   Thus \( \forall a, b \in Q, a * b = 1 + ab \in Q \)
   \( \therefore * \) is a binary operation.

7. If the b.o. on \( Z \) is defined by \( a * b = a + b + 5 \), find the identity element
   Ans: \( e = -5 \)

8. If \( * \) is defined by \( a * b = \frac{3ab}{7} \), \( \forall a, b \in R \), Find the identity element in \( R \) under \( * \)
   Ans: \( e = \frac{7}{3} \)

9. If \( * \) is defined by \( a * b = a^{b-1} \), find the identity if it exists.
   Ans: \( a^e = a^{e-1} \) and \( e^a = e^{a-1} \)
   i.e \( a^e \neq e^a \) Therefore identity does not exist
10. On $\mathbb{Q}^+$, $*$ is defined by $a * b = \frac{ab}{3}$, find the inverse of 3.
   Ans: $3^{-1} = 3$

11. In a group $(G, *)$, if $a * x = e, \forall a \in G$, find $x$.
   Ans: $x = a^{-1}$

12. If $*$ is defined by $a*b = a + b - ab$, solve the equation $6*x = 4$
   Ans: $6*x = 4$
   i.e. $6 + x - 6x = 4 \Rightarrow -5x = -2 \Rightarrow x = \frac{2}{5}$

13. Why the set of rationals does not form a group w.r.t multiplication?
   Ans: inverse law fails

14. Give an example of a finite group.
   Ans: $\{1, w, w^2\}$ under multiplication

15. Give an example of an infinite group.
   Ans: The set $\mathbb{Z}$ of integers under addition.

16. In the group $\{2, 4, 6, 8\}; \times \text{mod} \; 10,$ find the identity element
   Ans: $6^2 = 6 \implies 6$ is the identity [Or construct the composition table and find identity]

17. Find the inverse of 3 in the group $\{1, 3, 7, 9\}$ under $\times \text{mod} \; 10$ (O95)
   Ans: $e = 1; \; 3 \times 7 \equiv 1 \text{(mod10)} \implies 3^{-1} = 7$

18. In the group $G \{1, 3, 4, 5, 9\}$ under $\times \text{mod} \; 11,$ find the inverse of 5
   Ans: $e = 1; \; 5 \times 9 \equiv 1 \text{(mod11)} \implies 5^{-1} = 9$

19. In the group non-zero integers (mod 5) find $2^{-1}$ and $4^{-1}$
   Ans: $e = 1; \; 2 \times 3 \equiv 1 \implies 2^{-1} = 3$ and $4 \times 4 \equiv 1 \implies 4^{-1} = 4$

20. In the group $\{Z_{12}; + \text{mod} \; 6\}$ find $2 + 4^{-1} + 3^{-1}$
   Ans: $(2 + 6) + 3 = 1$

21. Name the b. o. under which the set of fourth roots of unity forms an abelian group.
   Ans: multiplication

   **PART-B**

22. If $a \ast b = \frac{a+b}{2}$ verify whether $*$ is associative.
Ans: \( a \cdot (b \cdot c) = a \cdot x \) where \( x = b \cdot c = \frac{b + c}{2} \)

\[
\begin{align*}
\frac{a + x}{2} &= \frac{1}{2} \left[ a + \frac{b + c}{2} \right] = \frac{2a + b + c}{4} \quad (1)
\end{align*}
\]

\( (a \cdot b) \cdot c = y \cdot c \) where \( y = a \cdot b = \frac{a + b}{2} \)

\[
\frac{y + c}{2} = \frac{1}{2} \left[ a + \frac{b + c}{2} + c \right] = \frac{a + b + 2c}{4} \quad (2)
\]

From (1) and (2) * is not associative

23. On I, the set of integers * is defined by \( a \cdot b = a + b - 7 \), find the identity and inverse of 14.

Ans: Let \( e \) be the identity

\( a \cdot e = a \Rightarrow a + e - 7 = a \Rightarrow e = 7 \)

Let \( a^{-1} \) be the inverse of \( a \)

By definition \( a \cdot a^{-1} = e \Rightarrow a + a^{-1} - 7 = 0 \Rightarrow a^{-1} = 14 - a \)

\( \therefore 14^{-1} = 14 - 14 = 0 \)

24. If \( G \) is group and \( a, b, c \in G \), then prove that \( a \cdot b = a \cdot c \Rightarrow b = c \)

25. In a group \( G \), prove that the equation \( a \cdot x = b \) has unique solution in \( G \).

26. P.T the identity element is unique in a group.

27. P.T the inverse of an element is unique in a group.

28. In a group \( G \), prove that \( (a \cdot b)^{-1} = b^{-1} \cdot a^{-1} \), \( \forall a, b \in G \)

29. Prove that a group of order 3 is abelian

30. If each of a group \( G \) is its own inverse then prove that \( G \) is abelian.

31. In a group \( G \), \( (ab)^2 = a^2b^2 \), \( \forall a, b \in G \), then prove that \( G \) is abelian.

Note: Questions 24 to 31 are all standard properties of groups. You get solution in all text books
32. Write the multiplication modulo 4 table for the set $Z_4=\{0,1,2,3\}$. Is $(Z_4, \times_4)$ a group?

$$
\begin{array}{c|cccc}
\times_4 & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 2 & 3 \\
2 & 0 & 2 & 0 & 2 \\
3 & 0 & 3 & 2 & 1 \\
\end{array}
$$

$Z_4$ is not a group under $\times_4$

33. Is $G=\{1, 2, 3, 4\}$ under $\otimes$ modulo 4 a group? Give reason

Ans: $2 \times_4 2 \neq 0 \in G$ i.e closure fails. $G$ is not a group.

$$
\begin{array}{c|cccc}
\times_4 & 1 & 2 & 3 & 4 \\
\hline
1 & 1 & 2 & 3 & 4 \\
2 & 2 & 0 & 2 & 0 \\
3 & 3 & 2 & 1 & 0 \\
4 & 0 & 0 & 0 & 0 \\
\end{array}
$$

34. In group of non–zero reals, $a \ast b = \frac{ab}{5}$. Solve for $x$ if $2 \ast (x \ast 5) = 10$

Ans: $2 \ast (x \ast 5) = 10 \Rightarrow 2 \ast \frac{x\times 5}{5} = 10$

i.e $2 \ast x = 10 \Rightarrow \frac{2x}{5} = 10 \Rightarrow x = 25$

35. Define a semi group. Give an example of a semi group which is not a group.

A nonempty set $G$ together with a binary operation $\ast$ is said to be a semigroup if $\ast$ satisfies closure and associative laws.

Ex: $(\mathbb{N}, +)$

36. Define a subgroup and give an example.

Ans: A non empty set $H$ is said to be subgroup of a group $(G, \ast)$ if

(i) $H \subseteq G$ (ii) $(H, \ast)$ is itself a group.

Ex: $\{1, -1\}$ is a subgroup of $\{1, -1, i, -i\}$ under multiplication.

37. Prove that the intersection of two sub groups of a group is again a subgroup
38. Prove that the set of integer \( \mathbb{Z} \) is an abelian group under the binary operation \( * \) defined by \( a * b = a + b - 5 \) for all \( a, b \in \mathbb{Z} \) and hence solve \( x * 3^{-1} = 2 \).

Soln:

Closure law:
Let \( a, b \in \mathbb{Z} \). Clearly, \( a + b - 5 \) is again an element of \( \mathbb{Z} \).
Thus \( a, b \in \mathbb{Z}, \ a * b = a + b - 5 \in \mathbb{Z} \).

Associative law: Let \( a, b, c \in \mathbb{Z} \).
Consider ,
\[
\begin{align*}
a * (b * c) &= a * x \quad \text{where } x = b * c = b + c - 5 \\
&= a + x - 5 = a + (b + c - 5) - 5 \\
&= a + b + c - 10
\end{align*}
\]
Again \( (a * b) * c \n\]
\[
\begin{align*}
&= y * c \quad \text{where } y = a * b = a + b - 5 \\
&= y + c - 5 = (a + b - 5) + c - 5 \\
&= a + b + c - 10
\end{align*}
\]
\[
\therefore \quad a * (b * c) = (a * b) * c
\]

Identity law: Let \( e \) be the identity
\[
a * e = a \Rightarrow a + e - 5 = a \Rightarrow e = 5
\]

Inverse law:
Let \( a^{-1} \) be the inverse of \( a \)
Then \( a * a^{-1} = e \)
\[
\Rightarrow a + a^{-1} - 5 = 5
\]
\[
\Rightarrow a^{-1} = 10 - a \quad \text{since } 3^{-1} = 10 - 3 = 7
\]

Commutative law: \( \forall a, b \in \mathbb{Z} \), we have
\[
a * b = a + b - 5 = b + a - 5 = b * a
\]

Hence \( (\mathbb{Z}, *) \) is an abelian group.
Consider \( x * 3^{-1} = 2 \ i.e. \ x * 7 = 2 \)
\[
\therefore \ x + 7 + 5 = 2 \Rightarrow x = -10
\]
39. If \( \mathbb{Q}_1 \) is the set of all rational number except 1 and \( * \) is a b.o. defined on \( \mathbb{Q}_1 \) by \( a * b = a + b - ab \),
Show that \( (\mathbb{Q}_1,*) \) is an abelian group.

   Hint: Associative law: \( a*(b*c)=(a*b)*c=a+b+c-ab-bc-ca+abc \)
   Identity element: \( e=0 \)
   Inverse law: \( a^{-1} = \frac{a}{a-1} \)

40. If \( \mathbb{Q}_-1 \) is the set of all rational number except \(-1\) and \( * \) is a b.o. defined on \( \mathbb{Q}_1 \) by \( a * b = a + b + ab \),
Show that \( (\mathbb{Q}_-1,*) \) is an abelian group.

   Hint: Associative law: \( a*(b*c)=(a*b)*c=a+b+c+ab+bc+ca+abc \)
   Identity element: \( e=0 \)
   Inverse law: \( a^{-1} = \frac{-a}{1+a} \)

41. In the set \( \mathbb{Q}_+ \) of all positive rational numbers define the operation \( * \) by \( a * b = ab/2 \), for all \( a, b \in \mathbb{Q}_+ \).
Prove that \( (\mathbb{Q}_+,*) \) is an abelian group

   Hint: Associative law: \( a*(b*c)=(a*b)*c=\frac{abc}{4} \)
   Identity element: \( e=2 \)
   Inverse law: \( a^{-1} = \frac{4}{a} \)

42. Show that the set \( G=\{\ldots,-3^3,5^2,5^1,5^0,5^1,5^2,5^3,\ldots\} \) is an abelian group under multiplication.

   Ans: Let \( a=5^p, b=5^q, c=5^r \) where \( p,q,r \in \mathbb{I} \). Then \( a,b,c \in G \)

   Closure law: \( a,b \in G \Rightarrow a \times b = 5^p \times 5^q = 5^{p+q} \in G \)

   Associative law: \( a \times (b \times c) = (5^p \times 5^q) \times 5^r = 5^{p+q+r} \)

   Also, \( (a \times b) \times c = (5^p \times 5^q) \times 5^r = 5^{p+q+r} \)

   \( \therefore a \times (b \times c) = (a \times b) \times c \)

   Identity law: Consider the element \( 5^0=1 \in G \).

   Clearly for every \( a \in G \), we have \( a \times 1 = 1 \times a = a \)

   Thus \( 5^0=1 \) is the identity

   Inverse law: Since the binary operation is multiplication \( a^{-1} = \frac{1}{a} \)

   \( i.e \) \( (5^p)^{-1} = \frac{1}{5^p} = 5^{-p} \in G \)

   Commutative law: \( \forall a,b \in G, a \times b = 5^p \times 5^q = 5^{p+q} = 5^q \times 5^p = b \times a \)

   Hence \( (G,*) \) is an abelian group.
43. Prove that $G = \{ \cos \theta + is\sin \theta : \theta \text{ is real} \}$ is an abelian group under multiplication.

Let $a = cis\alpha$, $b = cis\beta$, $c = cis\gamma$, Then $a, b, c \in G$.

Closure law: $a, b \in G \Rightarrow a \times b = cis\alpha \times cis\beta = cis(\alpha + \beta) \in G$.

Associative law: $a \times (b \times c) = cis\alpha \times (cis\beta \times cis\gamma) = cis(\alpha + \beta + \gamma)$.

Also, $(a \times b) \times c = (cis\alpha \times cis\beta) \times cis\gamma = cis(\alpha + \beta + \gamma)$.

$\therefore a \times (b \times c) = (a \times b) \times c$.

Identity law: Consider the element $cis0 = 1 \in G$.

Clearly for every $a \in G$, we have $a \times 1 = 1 \times a = a$.

Thus $cis0 = 1$ is the identity.

Inverse law: Since the binary operation is multiplication $a^{-1} = \frac{1}{a}$.

$i.e \ (cis\alpha)^{-1} = \frac{1}{cis\alpha} = cis(-\alpha) \in G$.

Commutative law: $\forall a, b \in G, \ a \times b = cis\alpha \times cis\beta = cis(\alpha + \beta)$.

$b \times a = cis\beta \times cis\alpha = cis(\beta + \alpha) = cis(\alpha + \beta)$.

Hence $(G, \ast)$ is an abelian group.

44. Prove that the set of multiples of 3 forms an abelian group under addition.

Ans: Let $G = \{3n : n \in I\} = \text{set of multiples of 3} = \{-9, -6, -3, 0, 3, 6, 9, \ldots\}$.

Let $a = 3p$, $b = 3q$, $c = 3r$, where $p, q, r \in I \therefore a, b, c \in I$.

Closure law. $\forall a, b \in G, \ a + b = 3p + 3q = 3(p + q) \in G$ closure law is valid in $G$.

Associative law: Since the addition of integers is associative.

$\forall a, b, c \in G, \ a + (b + c) = (a + b) + c$.

$\therefore$ associative law is valid in $I$.

Identity law: Since the b.o is addition the identity element is $0 \in G$.

Inverse law: Since the b.o is addition the inverse of $a$ is $-a$ because $a + (-a) = 0$.

Inverse of $a = -a$ i.e. inverse of $3p = -3p$.

Commutative $\forall a, b \in G, \ a + b = b + a$.

i.e. addition is commutative.
45. Prove that the set of all $2 \times 2$ matrices with elements of real numbers is an abelian group w.r.t. addition

Ans: Let $M$ be the set of all $2 \times 2$ matrices.

**Closure law.** Let $A = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ and $B = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \in M$

Then $A + B = \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix} \in M$.

**Associative law.** We know that the matrix addition is associative, i.e., $A + (B + C) = (A + B) + C$.

**Identity law:** Clearly, $O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in M$ is the identity and $A + O = O + A = A$ for all $A \in M$.

**Inverse law:** Since the binary operation is addition, the inverse of $A$ is $-A \in M$.

Hence, $M$ is a group under multiplication.

**Commutative law:** Clearly, $A + B = B + A$ for all $A, B \in M$.

$\therefore (M, +)$ is an abelian group.

46. Show that the set $M = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{R}, a, b \neq 0 \right\}$ of all matrices is a group under matrix multiplication.

**Closure law.** Let $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ and $B = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \in M$.

Then $AB = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} ac & 0 \\ 0 & bd \end{pmatrix} \in M$.

**Associative law.** We know that the matrix multiplication is associative.

$\therefore$ associative law is valid in $M$.

**Identity law:** Clearly, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in M$ is the identity and $A \times I = I \times A = A$ for all $A \in M$.

**Inverse law:** Let $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in M$. Clearly, $|A| = ab \neq 0$. Hence, $A$ is non-singular, $A^{-1}$ exists.
47. Prove that the set given by \(M = \left\{ \begin{pmatrix} x & x \\ x & x \end{pmatrix} \mid x \in R \neq 0 \right\} \) is an abelian group w.r.t. matrix multiplication.

48. Prove that \(\{1, 5, 7, 11\}\) is an abelian group under multiplication modulo 12

Ans: The composition table for G is

<table>
<thead>
<tr>
<th>4</th>
<th>12</th>
<th>5</th>
<th>7</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>5</td>
<td>7</td>
<td>11</td>
</tr>
<tr>
<td>12</td>
<td>12</td>
<td>5</td>
<td>7</td>
<td>11</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>1</td>
<td>11</td>
<td>7</td>
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<td>7</td>
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<td>11</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>11</td>
<td>11</td>
<td>7</td>
<td>5</td>
<td>1</td>
</tr>
</tbody>
</table>

Closure law: All the entries in the table are in G

Associative law: \(1 \times_{12} (5 \times_{12} 7) = 11\) Also \((1 \times_{12} 5) \times_{12} 7 = 11\)

\[\therefore 1 \times_{12} (5 \times_{12} 7) = (1 \times_{12} 5) \times_{12} 7\]

Identity law: 1 is the identity.

Inverse law: \(1^{-1} = 1\), \(5^{-1} = 5\), \(7^{-1} = 7\) and \(11^{-1} = 11\) i.e. each element of G has inverse

Therefore G is a group.

49. G= \(\{2, 4, 6, 8\}\) is a group under \(\times\) mod 10. Prepare the multiplication mod 10 table and hence find the identity

[3m]

50. Prove that the set \(H = \{1, 2, 4\}\) is a subgroup of the group \(G = \{1, 2, 3, 4, 5, 6\}\) under \(\times\) mod 7

Ans: \(H \subseteq G\)

The composition table for H is

<table>
<thead>
<tr>
<th>2</th>
<th>1</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

Closure law: All the entries in the table are in H

Associative law: \(1 \times_7 (2 \times_7 4) = 1\) Also \((1 \times_7 2) \times_7 4 = 1\)

\[
\therefore 1 \times_7 (2 \times_7 4) = (1 \times_7 2) \times_7 4
\]

Identity law: 1 is the identity.

Inverse law: \(1^1 = 1\), \(2^2 = 4\) and \(4^4 = 2\) i.e each element of H has inverse

Therefore H is itself a group and hence it is a subgroup.