GROUPS

PART-A

[1 MARK]

1. Define a binary operation

Ans: Let S be a nonempty set.* is said to be binary operation on S if

$$\forall\, a,b \in S, \quad a*b \in S \quad and \;\; is \;\; unique$$

2. a * b = $\sqrt{ab}$, Where a and b are the elements of R .Is * a binary operation on R ?

Ans: Let a=2, b=-7 then $a * b = 2 * (-7) = \sqrt{-14} \notin R$

$\therefore$ * is not a binary operation.

3. a * b = $\sqrt{ab}$, Where a and b are the elements of C . Is * a binary operation on C ?

Ans: Let a=1, b= i then $a * b = 1 * i = \sqrt{i} \;\in C \quad but\; \sqrt{i} \;\; is\; not\; unique$

$\therefore$ * is not a binary operation

4. In the set of all of natural numbers N, * is defined by a*b = 3a-4b . Is * a binary operation.

Ans: Let a=2, b=3 then $a * b = 2 * 3 = 3(2) - 4(3) = 6 - 12 = -6 \notin N$

$\therefore$ * is not a binary operation

5. In the set of nonnegative integers * is defined by a*b=a^b. Verify whether * is a binary operation or not.

Ans: Let a=0, b=0 then $a * b = 0 * 0 = 0^0 \; is\; not\; defined.$

$\therefore$ * is not a binary operation

6. If * is defined by a* b = 1 + ab , show that it is a. b.o. on Q .

Ans: Let a, b $\in$ Q. Product $a \cdot b$ of two rational numbers is again a rational number and 1+ab is also a rational number.

Thus $\forall\, a, b \;\in Q, a * b = 1 + ab \in Q$

$\therefore$ * is a binary operation

7. If the b.o $*$ on Z is defined by a * b = a + b +5 , find the identity element

Ans: e=-5

8. If * is defined by a * b = $\frac{3ab}{7}$, $\forall$ a, b$\in$R, Find the identity element in R under

Ans:e=7/3

9. If * is defined by $a * b = a^{b-1}$, find the identity if it exists.

Ans: a*e= a^{e-1} and e*a=e^{a-1}

i.e a*e $\neq$e*a Therefore identity does not exist

10. On Q^+ ,* is defined by $a * b = \frac{ab}{3}, find\ the\ inverse\ of\ 3.$

Ans:$3^{-1}=3$

11. In a group (G, *) , if a * x = e ,$\forall$ a$\in$ G, find x.

Ans: $x=a^{-1}$

12. If * is defined by a*b=a+b-ab, solve the equation 6*x=4

Ans: 6*x=4

i.e 6+x-6x=4$\Rightarrow$ -5x=-2$\Longrightarrow x = \frac{2}{5}$

13. Why the set of rationals does not form a group w.r.t multiplication ?

Ans: inverse law fails

14. Give an example of a finite group.

Ans: $\{1,w,w^2\}$ under multiplication

15. Give an example of a infinite group.

Ans: The set Z- of integers under addition.

16. In the group { 2, 4, 6, 8}; $\times$ mod 10 } , find the identity element

Ans: $6^2=6$ $\therefore$ 6 is the identity [Or construct the composition table and find identity]

17. Find the inverse of 3 in the group { 1, 3, 7, 9} under $\times$ mod 10 (O95)

Ans: e=1: $3\times7\equiv1(mod10)$ $\therefore$ $3^{-1}=7$

18. In the group G { 1, 3, 4, 5, 9} under $\times$mod 11 , find the inverse of 5

Ans: e=1: $5\times9\equiv1(mod11)$ $\therefore$ $5^{-1}=9$

19. In the group non-zero integers (mod 5) find 2^{-1} and 4^{-1}

Ans: e=1; $2\times3\equiv1$ $\therefore$ $2^{-1}=3$ and $4\times4\equiv1$ $\therefore$ $4^{-1}=4$

20. In the group { Z_6; + mod 6) find $2 + 4^{-1} + 3^{-1}$

Ans: $(2+_6 2) +_6 3=1$

21. Name the b. o. under which the set of fourth roots of unity forms an abelian group.

Ans: multiplication

PART-B

22. If $a * b = \frac{a+b}{2}$ verify whether * is associative.

Ans: a*(b*c)=a*x where x=b*c= $\frac{b+c}{2}$

$$= \frac{a+x}{2} = \frac{1}{2}\left[a+\frac{b+c}{2}\right] = \frac{2a+b+c}{4} \text{ --- (1)}$$

(a*b)*c=y*c where y=a*b= $\frac{a+b}{2}$

$$= \frac{y+c}{2} = \frac{1}{2}\left[\frac{a+b}{2}+c\right] = \frac{a+b+2c}{4} \qquad \text{---(2)}$$

From (1) and (2) * is not associative

23. On I, the set of integers * is defined by a*b=a+b-7, find the identity and inverse of 14.

Ans: Let e be the identity

a*e=a $\Rightarrow a+e-7=a \Rightarrow e=7$

Let a^{-1} be the inverse of a

By definition $a*a^{-1}=e \Rightarrow a+a^{-1}-7=7 \Rightarrow a^{-1}=14-a$

$\therefore 14^{-1} = 14-14 = 0$

24. if G is group and a, b, c $\in$ G, then prove that a*b = a*c$\Rightarrow$ b = c

25. In a group G, prove that the equation a *x = b has unique solution in G.

26. P.T the identity element is unique in a group

27. P.T the inverse of an element is unique in a group.

28. In a group G, prove that $(a*b)^{-1} = b^{-1}*a^{-1}$, $\forall$a, b$\in$ G

29. Prove that a group of order 3 is abelian

30. If each of a group G is its own inverse then prove that G is abelian.

31. In a group G, $(ab)^2 = a^2b^2$, $\forall$a, b $\in$ G, then prove that G is abelian .

Note: Questions 24 to 31 are all standard properties of groups. You get solution in all text books

32. Write the multiplication modulo 4 table for the set Z_4={0,1,2,3}. Is $(Z_4, \times_4)$ a group?

$\times_4$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

Z_4 is not a group under $\times_4$

33. Is G={ 1, 2, 3, 4} under $\otimes$ modulo 4 a group? Give reason

Ans: 2 $x_4$2=0$\notin G$ i.e closure fails. G is not a group.

$\times_4$	1	2	3	4
1	1	2	3	4
2	2	0	2	0
3	3	2	1	0
4	0	0	0	0

34. In group of non –zero reals , $a * b = \frac{ab}{5}$. Solve for x if 2 * (x * 5) = 10

Ans:$2 * (x * 5) = 10 \implies 2 * \frac{x \cdot 5}{5} = 10$

$i.e\ \ 2 * x = 10 \implies \frac{2x}{5} = 10 \implies x = 25$

35. Define a semi group . Give an example of a semi group which is not a group.

A nonempty set G together with a binary operation * is said to be a semigroup if * satisfies closure and associative laws.

Ex: (N, +)

36. Define a subgroup and give an example.

Ans: A non empty set H is said to be subgroup of a group (G,*) if

(i) H$\subset$ G (ii) (H,*) is itself a group.

Ex: {1,-1} is a subgroup of {1,-1,i,-i} under multiplication.

37. Prove that the intersection of two sub groups of a group is again a subgroup

PART-C

38. Prove that the set of integer Z is an abelian group under the binary operation $*$defined by a$*$ b = a+b–5 for all a, b $\in$ Z and hence solve x $*$ 3^{-1} =2.

Soln:

Closure law:

Let a,b $\in$Z. Clearly, a+b-5 is again an element of Z.

Thus a,b$\in$Z, a*b=a+b-5$\in$ Z

Associative law: Let a,b,c $\in$Z.

Consider ,

a*(b*c)= a*x where x=b*c=b+c-5

=a+x-5=a+(b+c-5)-5

= a+b+c-10

Again (a*b)*c

= y*c where y=a*b=a+b-5

=y+c-5=(a+b-5)+c-5

= a+b+c-10

$\therefore$ a*(b*c)=(a*b)*c

<u>Identity law:</u> Let e be the identity

a*e=a $\Rightarrow$a+e-5=a $\Rightarrow$ e=5

Inverse law:

Let a^{-1} be the inverse of a

Then $a*a^{-1}=e$

$\Rightarrow a+a^{-1}-5=5$

$\Rightarrow a^{-1}=10-a \in Z$ $\therefore 3^{-1} = 10 - 3 = 7$

Commutative law: $\forall$a,b $\in$Z,

we have

a*b=a+b-5=b+a-5=b*a

Hence (Z,*) is an abelian group

Consider x $*$ 3^{-1} =2 $i\ e\ \ x * 7 = 2$

Ie x+7+5=2$\Longrightarrow x = -10$

39. If Q_1 is the set of all rational number except 1 and * is a b.o. defined on Q_1 by a * b = a+ b -ab ,

Show that (Q_1 ,*) is an abelian group.

Hint: Associative law: a*(b*c)=(a*b)*c=a+b+c-ab-bc-ca+abc

Identity element: e=0

Inverse law: $a^{-1} = \frac{a}{a-1}$

40. If Q_{-1} is the set of all rational number except -1 and * is a b.o. defined on Q_1 by a * b = a+ b +ab ,

Show that (Q_{-1} ,*) is an abelian group.

Hint: Associative law: a*(b*c)=(a*b)*c=a+b+c+ab+bc+ca+abc

Identity element: e=0

Inverse law: $a^{-1} = \frac{-a}{1+a}$

41. In the set Q_+ of all positive rational numbers define the operation * by a * b = ab/2, for all a, b$\in Q_+$.

Prove that (Q_+ ,*) is an abelian group

Hint: Associative law: a*(b*c)=(a*b)*c=$\frac{abc}{4}$

Identity element: e=2

Inverse law: $a^{-1} = \frac{4}{a}$

42. Show that the set G= { ……5^{-3} ,5^{-2} ,5^{-1} ,5^0 5^1 ,5^2 ,5^3 .....} is an abelian group under multiplication.

Ans: Let a=5^p, b=5^q, c=5^r where p,q,r $\in$ I. Then a,b, c$\in$G

Closure law: a,b$\in$G$\Longrightarrow a \times b = 5^p \times 5^q = 5^{p+q} \in G$

Associative law: $a \times (b \times c) = 5^p \times (5^q \times 5^r) = 5^{p+q+r}$

Also, $(a \times b) \times c = (5^p \times 5^q) \times 5^r = 5^{p+q+r}$

$$\therefore\ a \times (b \times c) = (a \times b) \times c$$

Identity law: Consider the element 5^0=1$\in$ G.

Clearly for every a$\in$G, we have $a \times 1 = 1 \times a = a$

Thus 5^0=1 is the identity

Inverse law: Since the binary operation is multiplication $a^{-1} = \frac{1}{a}$

$$i.e\ (5^p)^{-1} = \frac{1}{5^p} = 5^{-p} \in G$$

Commutative law: $\forall\ a, b \in G,\ a \times b = 5^p \times 5^q = 5^{p+q} = 5^{q+p} = 5^q \times 5^p = b \times a$

Hence (G,*) is an abelian group.

43. Prove that G = { cos θ+ isin θ : θ is real }is an abelian group under multiplication

Let $a = cis\alpha,\ b = cis\beta, c = cis\gamma$, Then a,b, c∈G

Closure law: a,b∈G⟹ $a \times b = cis\alpha \times cis\beta = cis(\alpha + \beta) \in G$

Associative law: $a \times (b \times c) = cis\alpha \times (cis\beta \times cis\gamma) = cis(\alpha + \beta + \gamma)$

Also, $(a \times b) \times c = (cis\alpha \times cis\beta) \times cis\gamma = cis(\alpha + \beta + \gamma)$

$$\therefore\ a \times (b \times c) = (a \times b) \times c$$

Identity law: Consider the element cis0=1∈ G.

Clearly for every a∈G, we have $a \times 1 = 1 \times a = a$

Thus cis0=1 is the identity

Inverse law: Since the binary operation is multiplication $a^{-1} = \frac{1}{a}$

$$i.e\ (cis\alpha)^{-1} = \frac{1}{cis\alpha} = cis(-\alpha) \in G$$

Commutative law: $\forall\ a, b \in G,\ a \times b = cis\alpha \times cis\beta = cis(\alpha + \beta)$

$$b \times a = cis\beta \times cis\alpha = cis(\beta + \alpha) = cis(\alpha + \beta)$$

Hence (G,*) is an abelian group.

44. Prove that the set of multiples of 3 forms an abelian group under addition.

Ans: Let $G = \{3n : n \in I\} = set\ of\ multiples\ of\ 3 = \{\ldots -9, -6, -3, 0, 3, 6, 9 \ldots\}$

Let a=3p, b=3q, c=3r, where p,q,r ∈I ∴ a,b,c∈I

<u>Closure law</u>. $\forall a, b \in G,\quad a + b = 3p + 3q = 3(p + q) \in G$ closure law is valid in G

<u>Associative law</u> Since the addition of integers is associative

$$\forall a, b, c \in G,\quad a + (b + c) = (a + b) + c$$

∴ associative law is valid in I

<u>Identity law:</u> Since the b.o is addition the identity element is 0∈G

<u>Inverse law</u>: Since the b.o is addition the inverse of a is -a because a+(-a)=(-a)+a=

Inverse of a = -a i.e Inverse of 3p = -3p

<u>Commutative</u> $\forall a, b \in G,\quad a + b = b + a$

i.e addition is commutative

$\therefore$ (G,+) is an abelian group

45. Prove that the set of all 2×2 matrices with elements of real numbers is an abelian group w.r.t.addition

Ans :Let M be the set of all 2x2 matrices

<u>Closure law</u>. Let A= $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ and $B = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \in M$

Then A +B= $\begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix} \in M$ $\therefore$ closure law is valid in M

<u>Associative law</u> We know that the matrix addition is associative .i.e A+(B+C)=(A+B)+C

$\therefore$ associative law is valid in M

<u>Identity law:</u> Clearly O = $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in M$ is the identity and A+O=O+A=A $\forall A \in M$

<u>Inverse law:</u> Since the binary operation is addition inverse of A is -A $\in M$

Hence M is a group under multiplication.

<u>Commutative law</u> Clearly A+B=B+A $\forall A, B \in M$

$\therefore$ (M, +) is an abelian group.

46. Show that the set M= $\left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} a, b \in R, a, b \neq 0 \right\}$ of all matrices is a group under matrix multiplication

<u>Closure law</u>. Let A= $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ and $B = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \in M$

Then AB= $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} ac & 0 \\ 0 & bd \end{pmatrix} \in M$ $\therefore$ closure law is valid in M

<u>Associative law</u> We know that the matrix multiplication is associative.

$\therefore$ associative law is valid in M

<u>Identity law:</u> Clearly I = $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in M$ is the identity and A×I=I×A=A $\forall A \in M$

<u>Inverse law:</u> Let A= $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in M$ clearly $|A| = ab \neq 0$ Hence A is non singular, A^{-1} exists

$A^{-1} = \frac{1}{|A|} adjA = \frac{1}{ab}\begin{bmatrix} b & 0 \\ 0 & a \end{bmatrix} = \begin{bmatrix} 1/a & 0 \\ 0 & 1/b \end{bmatrix} \in M$

Hence M is a group under multiplication.

47. Prove tat the set given by M = $\left\{ \begin{pmatrix} x & x \\ x & x \end{pmatrix} \middle| \begin{matrix} x \in R \\ x \neq O \end{matrix} \right\}$ is an abelian group w.r.t. matrix multiplication.

48. Prove that ={ 1, 5, 7, 11} is an abelian group under multiplication modulo 12

Ans: The composition table for G is

X_{12}	1	5	7	11
1	1	5	7	11
5	5	1	11	7
7	7	11	1	5
11	11	7	5	1

Closure law: All the entries in the table are in G

Associative law: $1 \times_{12} (5 \times_{12} 7) = 11 \quad Also\ (1 \times_{12} 5) \times_{12} 7 = 11$

$$\therefore 1 \times_{12} (5 \times_{12} 7) = (1 \times_{12} 5) \times_{12} 7$$

Identity law: 1 is the identity.

Inverse law: $1^{-1} = 1$, $5^{-1} = 5$, $7^{-1} = 7$ and $11^{-1} = 11$ i.e each element of G has inverse

Therefore G is a group.

49. G= { 2, 4, 6, 8} is a group under $\times$ mod 10. Prepare the multiplication mod 10 table and hence find the identity

[3m]

50. Prove that the set H= {1,2,4} is a subgroup of the group G = {1, 2, 3, 4, 5, 6} under $\times$ mod 7 [3m]

Ans: $H \subset G$

The composition table for H is

X_7	1	2	4

1	1	2	4
2	2	4	1
4	4	1	2

Closure law: All the entries in the table are in H

Associative law: $1 \times_7 (2 \times_7 4) = 1 \quad Also\ (1 \times_7 2) \times_7 4 = 1$

$$\therefore 1 \times_7 (2 \times_7 4) = (1 \times_7 2) \times_7 4$$

Identity law: 1 is the identity.

Inverse law: $1^{-1} = 1$, $2^{-1} = 4$ and $4^{-1} = 2$ i.e each element of H has inverse

Therefore H is itself a group and hence it is a subgroup.