KEA

MATHEMATICS

Applications of Derivatives

MATHEMATICS

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MATHEMATICS

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MATHEMATICS

MATHEMATICS

1) The points on the curve $y = x^3 - 3x$

where the tangents drawn are parallel to the x - axis are

(1) (1, 2) and (- 1, 2)

(2) (1, - 2) and (- 1, 2)

(3) (1, 2) and (- 1, - 2)

(4) (1, - 2) and (- 1, 2)

MATHEMATICS

$$y = x^3 - 3x$$

$$\frac{dy}{dx} = 0 \Rightarrow 3x^2 - 3 = 0 \Rightarrow 3(x^2 - 1) = 0$$

$$\Rightarrow x = \pm 1$$

$$\text{when } \; x = 1, \qquad y = 1 - 3 = -2$$

$$\text{when } \; x = -1, \qquad y = -1 + 3 = 2$$

$\therefore$ the points are (1, - 2) and (- 1, 2)

Choice (2) is the correct answer

MATHEMATICS

2) The tangent to the curve $y = 2x^2 - 3x + 1$

at (2, 3) on it is

(1) Parallel to $5x - y - 1 = 0$ **(2)** Parallel to $y = 3x + 5$

(3) Perpendicular to $5x - y + 3$ **(4)** Parallel to the x - axis

MATHEMATICS

$$y=2x^2-3x+1$$

$$\frac{dy}{dx}=4x-3$$

$$\left(\frac{dy}{dx}\right)_{(2,3)}=8-3=5$$

Hence, the tangent is parallel to $5x - y - 1 = 0$

Choice (1) is the correct answer

MATHEMATICS

3) The equation of the normal to the curve $y = ae^{\frac{x}{b}}$

where it crosses the y – axis is

(1) $bx - ay = a^2$

(2) $ax + by = ab$

(3) $ax - by = ab$

(4) $bx + ay = a^2$

MATHEMATICS

$$y = ae^{\frac{x}{b}}$$

Put x = 0, y = a. ∴ (0, a) is the point

$$\left(\frac{dy}{dx}\right) = \frac{a}{b}e^{\frac{x}{b}} \qquad \therefore \left(\frac{dy}{dx}\right)_{(0,a)} = \frac{a}{b}$$

Slope of the normal = $\frac{-b}{a}$

Eqn: $y - a = \frac{-b}{a}(x - 0) \quad \Rightarrow bx + ay = a^2$

Choice (4) is the correct answer

MATHEMATICS

4) The tangent drawn to the curve $x = e^t \cos t$, $y = e^t \sin t$ at t = 0, makes an angle with the x – axis equal to

(1) 0

(2) $\frac{\pi}{2}$

(3) $\frac{\pi}{4}$

(4) $\frac{\pi}{3}$

MATHEMATICS

$$\frac{dy}{dx}=\frac{e^t(\sin t+\cos t)}{e^t(\cos t-\sin t)}$$

$$\frac{dy}{dx}=\frac{\cancel{e^t}(\sin t+\cos t)}{\cancel{e^t}(\cos t-\sin t)}$$

$$\left(\frac{dy}{dx}\right)_{t=0}=1$$

$\therefore$ Angle made by the tangent with the x – axis is $\frac{\pi}{4}$

Choice (3) is the correct answer

5) If the tangent to the ellipse $\frac{x^2}{16}+\frac{y^2}{4}=1$ at the point ‘θ’ on it is normal to the circle $x^2+y^2-16x=0$, then θ =

(1) $\frac{\pi}{6}$

(2) $\frac{\pi}{3}$

(3) $\frac{\pi}{4}$

(4) $\frac{\pi}{2}$

MATHEMATICS

Point ‘θ’ is (4 cos θ, 2 sinθ)

Equation of tangent at this point is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$

i.e. $\frac{x(\cancel{4}\cos\theta)}{\cancel{16}\ 4} + \frac{y(\cancel{2}\sin\theta)}{\cancel{4}\ 2} = 1$

$$\Rightarrow x\cos\theta + 2y\sin\theta = 4$$

Centre (8, 0) is a point on this. $\therefore$ 8 cosθ = 4

$$\Rightarrow \cos\theta = \frac{1}{2}$$

$$\therefore \quad \theta = \frac{\pi}{3}$$

Choice (2) is the correct answer

KEA

MATHEMATICS

6) The curves $y = x^3$ and $y = x^2 + x - 1$ at the point (1, 1)

(1) Cut orthogonally

(2) Touch each other

(3) Intersect at angle $\frac{\pi}{4}$

(4) Intersect at angle $\frac{\pi}{6}$

MATHEMATICS

$$y = x^3 \qquad\qquad y = x^2 + x - 1$$

$$\frac{dy}{dx} = 3x^2 \qquad\qquad \frac{dy}{dx} = 2x + 1$$

$$m_1 = \left(\frac{dy}{dx}\right)_{(1,1)} = 3 \qquad\qquad m_2 = \left(\frac{dy}{dx}\right)_{(1,1)} = 3$$

$$m_1 = m_2$$

$\therefore$ The two curves touch each other

Choice (2) is the correct answer

MATHEMATICS

7) If θ is the acute angle between $x^2 + y^2 = 4x$ and $x^2 + y^2 = 8$ at (2, 2), then sin θ is equal to

(1) 1

(2) $\frac{\sqrt{3}}{2}$

(3) $\frac{1}{\sqrt{2}}$

(4) $\frac{1}{2}$

MATHEMATICS

$$x^2 + y^2 = 4x$$

$$2x + 2y\frac{dy}{dx} = 4$$

$$\Rightarrow \frac{dy}{dx} = \frac{2-x}{y}$$

$$m_1 = \left(\frac{dy}{dx}\right)_{(2,2)} = 0$$

Diff. w.r.t x

$$x^2 + y^2 = 8$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-x}{y}$$

$$m_2 = \left(\frac{dy}{dx}\right)_{(2,2)} = -1$$

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| = \left|\frac{-1-0}{1+0}\right| = 1$$

$$\Rightarrow \theta = \frac{\pi}{4} \qquad \therefore \ \sin\theta = \frac{1}{\sqrt{2}}$$

Choice (3) is the correct answer

MATHEMATICS

8) If the curves $y^2 = 4x$ and $xy = K$ intersect at right angles, then K^2 =

(1) 32

(2) 16

(3) 8

(4) 64

MATHEMATICS

$y^2 = 4x \longrightarrow (1)$ $xy = K \longrightarrow (2)$

$$\frac{dy}{dx} = \frac{2}{y}$$

$$x\frac{dy}{dx} + y = 0 \Rightarrow \frac{dy}{dx} = \frac{-y}{x}$$

Since the curves cut orthogonally, $\left(\frac{2}{y}\right)\left(\frac{-y}{x}\right) = -1$

$$\Rightarrow x = 2$$

Substitute $x = 2$ in (1) $y^2 = 8$

From (2) $x^2y^2 = K^2$ $\Rightarrow K^2 = (4)(8) = 32$

Choice (1) is the correct answer

MATHEMATICS

9) The lengths of the sub-tangent and sub-normal to the curve $x^2 + xy + y^2 = 7$ at (1, - 3) are respectively

(1) 15 and $\frac{5}{3}$

(2) 5 and $\frac{3}{5}$

(3) 15 and $\frac{1}{5}$

(4) 15 and $\frac{3}{5}$

MATHEMATICS

$$x^2 + xy + y^2 = 7$$

Diff. w.r.t x $$2x + x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-2x - y}{(x+2y)}$$

$$\left(\frac{dy}{dx}\right)_{(1,-3)} = \frac{-2+3}{1-6} = \frac{-1}{5}$$

$$ST = \frac{y}{\left(\frac{dy}{dx}\right)} = (-3)(-5) = 15$$

$$SN = y\frac{dy}{dx} = \frac{3}{5}$$

Choice (4) is the correct answer

MATHEMATICS

10) For the curve $y^2 = 4ax$ at any point

(1) S.T is a constant and S.N $\propto$ y^2

(2) S.T $\propto$ y and S. N is a constant

(3) S.N is a constant and S.T $\propto$ x

(4) Both S. T and S. N are constants

MATHEMATICS

$$y^2 = 4ax$$

$$\Rightarrow 2y\frac{dy}{dx} = 4a$$

$$\Rightarrow y\frac{dy}{dx} = 2a$$ which is a constant

$$y^2 = 4ax$$

Taking log $2\log y = \log 4a + \log x$

Diff. w.r.t x $2.\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}$

i.e $2.\frac{1}{ST} = \frac{1}{x}$

$$\Rightarrow \quad S.T = 2x$$

Choice (3) is the correct answer

MATHEMATICS

11) For the curve $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$

S. T = S. N at the point $\theta =$

(1) $\frac{\pi}{4}$

(2) $\frac{\pi}{2}$

(3) π

(4) $\frac{\pi}{6}$

MATHEMATICS

$$ST = SN \;\Rightarrow\; \left(\frac{dy}{dx}\right)^2 = 1 \qquad \Rightarrow\; \frac{dy}{dx} = \pm\ 1$$

From the equations $x = a\,(\theta + \sin\theta)$ and $y = a\,(1 - \cos\theta)$

$$\frac{dy}{dx} = \frac{\cancel{a}(\sin\theta)}{\cancel{a}(1+\cos\theta)}$$

$$\text{when }\ \theta = \frac{\pi}{2}\quad ,\quad \frac{dy}{dx} = 1$$

Choice (2) is the correct answer

MATHEMATICS

12) A stone projected vertically upwards moves a distance S metre in time t second given by $S = 12t - 2.4t^2$

The time taken by the stone in second to reach the greatest height and the greatest height in metre attained by the stone are respectively

(1) 2.5 and 30

(2) 2.0 and 15

(3) 2.5 and 25

(4) 2.5 and 15

MATHEMATICS

$$S = 12t - 2.4t^2$$

$$\frac{dS}{dt} = 0$$

$$\Rightarrow 12 - 4.8t = 0$$

$$\Rightarrow t = \frac{12}{4.8} = \frac{120}{48} = \frac{5}{2}$$

When $t = \frac{5}{2}$

$$S = 12\left(\frac{5}{2}\right) - 2.4\left(\frac{25}{4}\right) = 30 - 15 = 15$$

Choice (4) is the correct answer

MATHEMATICS

13) If each side of an equilateral triangle is increasing at the rate of 4 cm/sec, then the rate at which its area is increasing when the side is 6 cm in sq. cm/sec unit is

(1) $6\sqrt{3}$

(2) $12\sqrt{3}$

(3) $4\sqrt{3}$

(4) $24\sqrt{3}$

MATHEMATICS

$$A = \frac{\sqrt{3}}{4}L^2$$

$$\frac{dA}{dL} = \frac{\sqrt{3}}{4}(2L)\frac{dL}{dt}$$

$$= \frac{\sqrt{3}}{\cancel{4}}(2)(6)\cancel{(4)} = 12\sqrt{3}$$

Choice (2) is the correct answer

MATHEMATICS

14) If the distance 'S' travelled by a particle is proportional to the square root of its velocity, then its acceleration is

(1) A constant

(2) $\propto S^2$

(3) $\propto S^3$

(4) $\propto \frac{1}{S^3}$

MATHEMATICS

$$S = K\sqrt{v}$$

$$\Rightarrow S^2 = K^2 v \Rightarrow 2S\frac{dS}{dt} = K^2\frac{dv}{dt}$$

$$\therefore \frac{dv}{dt} = \frac{2Sv}{K^2} = \frac{2S}{K^2}\left(\frac{S^2}{K^2}\right)$$

$$= (\text{constant})\, S^3$$

$$\Rightarrow \text{acceleration} \propto S^3$$

Choice (4) is the correct answer

MATHEMATICS

15) The volume of a sphere is increasing at the rate of 4 π cc/sec. The rate at which its radius is increasing, when its surface area is 64 π cc in cm/sec unit is

(1) $\frac{1}{8}$

(2) $\frac{1}{16}$

(3) $\frac{1}{4}$

(4) 16

MATHEMATICS

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = \frac{4}{\cancel{3}}\pi \cancel{3}r^2 \frac{dr}{dt}$$

Given $4\pi r^2 = 64\pi$

$$4\pi = 64\pi \frac{dr}{dt}$$

$$\therefore \frac{dr}{dt} = \frac{1}{16}$$

Choice (2) is the correct answer

MATHEMATICS

16) The stationary points of the function $x^3 - 3x^2 - 9x + 1$ are

(1) x = 3 and x = 1

(2) x = - 3 and x = - 1

(3) x = 3 and x = - 1

(4) x = - 3 and x = 1

MATHEMATICS

$$y = x^3 - 3x^2 - 9x + 1$$

$$\frac{dy}{dx} = 3x^2 - 6x - 9$$

$$= 3(x^2 - 2x - 3)$$

Clearly when x = - 1, and x = 3, $\frac{dy}{dx} = 0$

Choice (3) is the correct answer

MATHEMATICS

17) The maximum value of xe^{-x} is

(1) e

(2) $\frac{1}{e}$

(3) 1

(4) e^2

MATHEMATICS

$$f(x) = xe^{-x}$$

$$f'(x) = 0 \Rightarrow (-x+1)e^{-x} = 0$$

$$\Rightarrow x = 1$$

$$f(1) = e^{-1} = \frac{1}{e}$$

Choice (2) is the correct answer

MATHEMATICS

18) The area of a circular sector is 16 sq. units. The radius Of the sector for which the perimeter is minimum is

(1) 4

(2) 8

(3) 2

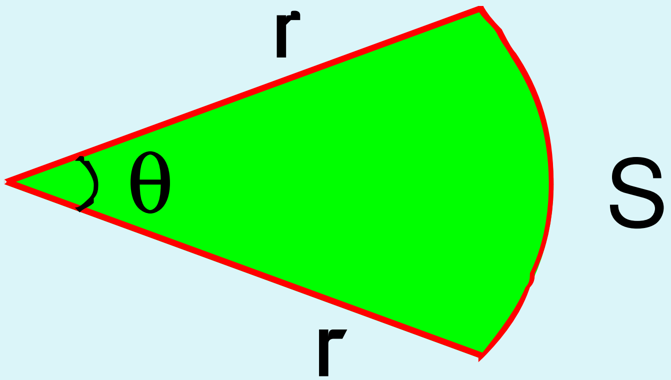
(4) 6

MATHEMATICS

given $$\frac{1}{2}r^2\theta = 16$$

$$\Rightarrow r\theta = \frac{32}{r}$$

Perimeter $$P = 2r + r\theta$$



$$P = 2r + \frac{32}{r}$$

$$\frac{dP}{dr} = 0 \Rightarrow 2 - \frac{32}{r^2} = 0 \Rightarrow r^2 = 16$$

or $r = 4$

Choice (1) is the correct answer

MATHEMATICS

19) If x = 1 and x = - 2 are points of minima and maxima respectively of a function f(x) and $f'(0) = -2$. Then $f'(2) =$

(1) - 4

(2) 2

(3) 4

(4) 8

MATHEMATICS

At x = 1 and x = - 2 $\frac{dy}{dx} = f'(x) = 0$

$$\therefore f'(x) = k(x-1)(x+2)$$

$$f'(0) = -2 \Rightarrow k = 1$$

$$\therefore f'(x) = (x-1)(x+2)$$

$$f'(2) = 4$$

Choice (3) is the correct answer

MATHEMATICS

20) The maximum value of $\sqrt{3}\cos x + \sin x$ is

(1) $\sqrt{3}$

(2) 2

(3) 1

(4) $\sqrt{3}+1$

MATHEMATICS

$$\sqrt{3}\cos x + \sin x = 2\left(\frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x\right)$$

$$= 2\cos\left(x - \frac{\pi}{6}\right)$$

$$-1 \le \cos\theta \le 1$$

$$\therefore \quad -1 \le \cos\left(x - \frac{\pi}{6}\right) \le 1$$

$$\therefore \quad -2 \le 2\cos\left(x - \frac{\pi}{6}\right) \le 2$$

Choice (2) is the correct answer

MATHEMATICS

21) The maximum value of $15-\sqrt{14+3\cos x-4\sin x}$ is

(1) 15

(2) 12

(3) 10

(4) 5

MATHEMATICS

Minimum value of 3 cos x – 4 sin x is – 5

∴ Maximum value of given expression

$$15 - \sqrt{14 - 5}$$

$$15 - 3 = 12$$

Choice (2) is the correct answer

MATHEMATICS

22) If the maximum value of $a\cos x + \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$ is 5,

then the value of a =

(1) 1 or – 3

(2) 1 or 3

(3) – 1 or 3

(4) – 1 or – 3

MATHEMATICS

$$a\cos x+\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)=a\cos x+\sqrt{2}\left(\sin x\left(\frac{1}{\sqrt{2}}\right)+\cos x\left(\frac{1}{\sqrt{2}}\right)\right)$$

$$=a\cos x+\sin x+\cos x$$

$$=(a+1)\cos x+\sin x$$

Maximum value of this is $\sqrt{(a+1)^2+1}=5$ (Given)

$$\Rightarrow (a+1)^2+1=5$$

$$(a+1)^2=4$$

$$a+1=\pm 2$$

$$\therefore\ a=1 \text{ or } -3$$

Choice (1) is the correct answer

MATHEMATICS

23) The minimum and maximum values of $\sin^4 x + \cos^4 x$ are

(1) $\frac{1}{2}$ and $\frac{3}{2}$

(2) $\frac{1}{2}$ and 1

(3) 1 and $\sqrt{2}$

(4) $\frac{1}{2}$ and $\sqrt{2}$

MATHEMATICS

$$\sin^4 x + \cos^4 x = \left(\sin^2 x + \cos^2 x\right)^2 - 2\sin^2 x\cos^2 x$$

$$= 1 - \frac{1}{2}(2\sin x\cos x)^2$$

$$= 1 - \frac{1}{2}\sin^2 2x$$

Max. value of $\sin^2 2x$ is 1 and

Min. value is 0

$\therefore$ Minimum value of the given expression is $1 - \frac{1}{2} = \frac{1}{2}$

Maximum value = 1 – 0 = 1

Choice (2) is the correct answer

MATHEMATICS

24) Sum of two positive numbers is 48. The maximum value of their product is

(1)	320	**(2)**	560
(3)	576	**(4)**	380

MATHEMATICS

When sum of two numbers is given, say M, we know that the product is Maximum only when the numbers are equal

i.e $\frac{M}{2}$ and $\frac{M}{2}$

Therefore the maximum product is $\left(\frac{M}{2}\right)^2$

In this case it is $\left(\frac{48}{2}\right)^2 = 576$

Choice (3) is the correct answer

MATHEMATICS

25) If α, β and 2 are the roots of the equation $x^3 - 22x^2 + bx - c = 0$ then the maximum value of 'c' is

(1) 400

(2) 200

(3) 350

(4) 600

MATHEMATICS

Since α, β and 2 are the roots of the given equation

$\alpha + \beta + 2 = 22$ and $(\alpha\beta)(2) = c$

$\alpha + \beta = 20$

$2\alpha\beta = c$

$\alpha\beta = \frac{c}{2}$

The product α β is maximum only when α = β = 10

∴ Max. value of α β = (10) (10)

⇒ Max. value of $\frac{c}{2} = 100$

$\therefore \; c_{max} = 200$

Choice (2) is the correct answer

MATHEMATICS